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Permutation invariant algebras, a Fock space realization and the Calogero model

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Abstract. We study permutation invariant oscillator algebras and their Fock space representations using three equivalent techniques, i.e. (i) a normally ordered expansion in creation and annihilation operators, (ii) the action of annihilation operators on monomial states in Fock space and (iii) Gram matrices of inner products in Fock space. We separately discuss permutation invariant algebras which possess hermitean number operators and permutation invariant algebras which possess non-hermitean number operators. The results of a general analysis are applied to the S_M-extended Heisenberg algebra, underlying the M-body Calogero model. Particular attention is devoted to the analysis of Gram matrices for the Calogero model. We discuss their structure, eigenvalues and eigenstates. We obtain a general condition for positivity of eigenvalues, meaning that all norms of states in Fock space are positive if this condition is satisfied. We find a universal critical point at which the reduction of the physical degrees of freedom occurs. We construct dual operators, leading to the ordinary Heisenberg algebra of free Bose oscillators. From the Fock-space point of view, we briefly discuss the existence of a mapping from the Calogero oscillators to the free Bose oscillators and vice versa.

1 Introduction

The classical and quantum integrable model of M interacting particles on a line, introduced by Calogero [1], has been intensively studied during the past few years. The model and its generalizations [2] are connected with a number of physical problems, ranging from condensed matter physics [3] to gravity and black-hole physics [4]. The algebraic structure of the Calogero model and its successors, studied earlier using group theoretical methods [5], has recently been reconsidered by a number of authors in the framework of the S_M-extended Heisenberg algebra [6].

Apart from its particular realization, the S_M-extended Heisenberg algebra is basically *a multi-mode oscillator* algebra with *permutation invariance*. The general techniques for analyzing such a class of oscillator algebras were developed earlier in a series of papers [7,8].

In the present paper we apply these techniques to the Calogero model in its operator formulation. We start our analysis with the two-body Calogero model. On the algebraic grounds, this model is described by a particular class of deformed single-mode oscillator algebras which were treated in a unified manner in [9].

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In Sect. 2, we describe a single-mode algebra underlying the two-body Calogero model. We construct a mapping from this algebra to the ordinary Bose algebra. We also present the number operator N and the exchange operator K as an (infinite) series in the creation and annihilation operators. At the end of Sect. 2 we construct an algebra that is dual to the original algebra of the two-body Calogero model.

In Sect. 3 we discuss general multi-mode oscillator algebras with permutation invariance. We describe two distinct classes of these algebras:

(i) algebras which possess well-defined *hermitean* number operators (i.e. transition number operators N_{ij}, partial number operators $N_{ii} \equiv N_i$ and the total number operator $N \equiv \sum N_i$), and
(ii) algebras which possess well-defined number operators but for which only the total number operator N is hermitean.

The analysis of these algebras is performed in three equivalent ways: using

(i) a normally ordered expansion in creation and annihilation operators,
(ii) the action of annihilation operators on monomial states in Fock space, and
(iii) Gram matrices of scalar products in Fock space. We conclude Sect. 3 with a discussion of the general

structure of transition number operators N_{ij} and exchange operators K_{ij}.

The ideas developed in the preceding sections are applied to the many-body Calogero model in Sect. 4. The algebra underlying the many-body Calogero model (S_M-extended Heisenberg algebra) is discussed along the lines described in Sect. 3. Special attention is devoted to the analysis of Gram matrices and to the construction of number operators and exchange operators as an infinite series in the creation and annihilation operators. We also find that the S_M-extended Heisenberg algebra can be defined as a generalized triple operator algebra. Generalizing the construction given in Sect. 2 to the multi-mode case, we define and investigate the structure of dual algebras. Section 4 ends with a short discussion of mappings from the Calogero algebra to the set of free bosonic oscillators. Section 5 is a short summary.

2 Two-body Calogero model and deformed single-mode oscillator algebras

Particular aspects of single-mode deformed oscillator algebras were studied by a number of authors, starting with the seminal papers of [10]. A unified view of deformed single-mode oscillator algebras was proposed in [9,11].

Basically, these algebras are generated by a set of generators $\mathcal{G}$, involving annihilation (a) and creation ($a^\dagger$) operators, together with the well-defined number operator N:

$$\mathcal{G} := \{1, a, a^\dagger, N\},$$
$$(a)^\dagger = a^\dagger, \quad N = N^\dagger.$$

The following commutation relations hold:

$$\begin{aligned} &[N,a] = -a, \quad [N,a^\dagger] = a^\dagger, \\ &[N, a^\dagger a] = [N, aa^\dagger] = 0, \\ &aa^\dagger - q a^\dagger a = G(N), \end{aligned} \tag{1}$$

where $q \in \mathbf{R}$ and $G(N)$ is a hermitean, analytic function of the number operator. The vacuum conditions are $a|0\rangle = 0$ and $N|0\rangle = 0$, with $\langle 0|0\rangle = 1$. Since $[N, a^\dagger a] = [N, aa^\dagger] = 0$, we can write

$$\begin{aligned} a^\dagger a &= \phi(N), \\ aa^\dagger &= \phi(N+1), \end{aligned} \tag{2}$$

where $\phi(N) \geq 0$ is some function of the number operator. Several examples of algebras that belong to the class (1) and their corresponding functions $\phi(N)$ are given in [9].

Here we want to discuss a variant of the algebra (1), with $G(N) = 1 + 2\nu K$ and $q = 1$, namely

$$\begin{aligned} aa^\dagger - a^\dagger a = 1 + 2\nu K, \quad \nu \in \mathbf{R}, \\ K = (-)^N, \quad Ka = -aK. \end{aligned} \tag{3}$$

In this equation K is the exchange operator (see Sects. 3 and 4) which here acts simply as a parity operator that separates the set of excited states $|n\rangle \propto a^{\dagger n}|0\rangle$ into even and odd subspaces. For $\nu > -1/2$, the algebra (3) possesses unitary infinite-dimensional representations. This algebra is known as the Calogero–Vasiliev algebra [6] (also termed the deformed Heisenberg algebra with reflection [12]) and provides an algebraic formulation of the two-particle Calogero model [1] described by the Hamiltonian (x and p are the relative coordinate and momentum, respectively)

$$2H = p^2 + x^2 + \frac{\nu(\nu-1)}{x^2} K, \tag{4}$$

which reduces to

$$2H = \{a, a^\dagger\} \tag{5}$$

after the identification

$$\begin{aligned} \sqrt{2}a &= x + \mathrm{i}p - \frac{\nu}{x}K, \\ \sqrt{2}a^\dagger &= x - \mathrm{i}p + \frac{\nu}{x}K. \end{aligned}$$

Remark 1. Generalizations of the algebra (3) have been investigated in [13,14] and its connection to nonlinear parabosonic (parafermionic) supersymmetry has been described in [15].

As we have already described in [7,9], the analysis of the general (deformed) oscillator algebras could be carried out in three completely equivalent ways.

One can express $aa^\dagger$ as a normally ordered expansion:

$$aa^\dagger = 1 + \sum_{k\geq 1} \alpha_k a^{\dagger k} a^k. \tag{6}$$

Alternatively, one can start the analysis by using the action of the annihilation operator on the states in Fock space:

$$a a^{\dagger m}|0\rangle = \phi(N+1) a^{\dagger m-1}|0\rangle. \tag{7}$$

The third way is to examine the vacuum matrix elements (Gram matrix)

$$A_{m,n} = \langle 0|a^m a^{\dagger n}|0\rangle = [\phi(n)]!\, \delta_{mn}. \tag{8}$$

These approaches are rather simple for the single-mode oscillators but become very powerful for the analysis of multi-mode (deformed) oscillator algebras (see Sects. 3 and 4).

Now, we show how the normally ordered expansion (6) works for the Calogero–Vasiliev algebra (3).

First, we calculate the function $\phi(N)$; it reads

$$\phi(N) = N + \nu(1 + (-)^{1+N}). \tag{9}$$

Knowing $\phi(N)$, we recursively calculate the coefficients α_k (see (6)) for the algebra (3):

$$\alpha_k = \frac{\phi(k+1) - 1 - \sum_{m=1}^{k-1} \alpha_m \phi(k)\cdots\phi(k+1-m)}{[\phi(k)]!},$$
$$\forall \phi(k) \neq 0.$$

Similarly, we can expand the operators K and N in an infinite series in the operators a and $a^\dagger$, i.e.

$$K = 1 + \sum_{k\geq 1} \beta_k a^{\dagger k} a^k. \tag{10}$$

Using relations (3) and $Ka^{\dagger n}|0\rangle = (-)^n a^{\dagger n}|0\rangle$, one can recursively calculate the coefficients β_k as

$$\beta_k = \frac{[(-)^k - 1]}{\phi(k)!} - \sum_{m=1}^{k-1} \beta_m \frac{1}{\phi(m-1)!}.$$

The expansion of the number operator N reads

$$N = a^\dagger a + \sum_{n\geq 2} \gamma_n (a^\dagger)^n (a)^n, \tag{11}$$

where

$$\gamma_n = \frac{n - \sum_{k=1}^{n-1} \gamma_k \phi(n) \cdots \phi(n+1-k)}{[\phi(n)]!}.$$

Note that $\gamma_n = 0$ for $\phi(n) = 0$, $\phi(n-1) \neq 0$.

Notice also that, knowing α_k, β_k and γ_n, we can obtain $\phi(n)$ from the same recurrent relation.

As we elaborated in [9], there exists a simple mapping of the general deformed algebra (1) to the ordinary Bose algebra $[b, b^\dagger] = 1$. This mapping is of the form

$$a = b\sqrt{\frac{a^\dagger a}{N}} \equiv b\sqrt{\frac{\phi(N)}{N}}. \tag{12}$$

The inverse mapping exists if $\phi(N) \neq 0$, i.e. $\phi(N) > 0$, $\forall N$. Using $\phi(N)$, and (9), associated with the algebra (3), we obtain

$$a = \begin{cases} b, & n = \text{even}, \\ b\sqrt{\dfrac{N+2\nu}{N}}, & n = \text{odd}. \end{cases}$$

It is also possible to map the operators a and $a^\dagger$ to the Bose operators b and $b^\dagger$, using an expansion of the form

$$a = \left(\sum_{k\geq 0} c_k b^{\dagger k} b^k \right) \cdot b. \tag{13}$$

Comparing (13) with (3), one can recursively calculate the coefficients c_k as

$$c_k = \frac{1}{k!} \left[\sqrt{\frac{\phi(k+1)}{(k+1)}} + \sum_{m=1}^{k} (-)^k \begin{pmatrix} k \\ m \end{pmatrix} \sqrt{\frac{\phi(k+1-m)}{(k+1-m)}} \right].$$

For our further purposes (see Sect. 4), it is convenient to define a new operator $\tilde{a}$ in a sense dual to $(a, a^\dagger)$, such that

$$[\tilde{a}, a^\dagger] = 1, \quad \tilde{a}|0\rangle = 0. \tag{14}$$

The connection is

$$\tilde{a} = a\frac{N}{a^\dagger a} \equiv a\frac{N}{\phi(N)}, \quad \phi(N) > 0, \quad \forall N, \tag{15}$$

where N can be realized as in (11), or as

$$N = \frac{1}{2}\{a^\dagger, a\} - \left(\nu + \frac{1}{2}\right).$$

Using $\phi(N)$, (9), we find

$$\tilde{a} = \begin{cases} a, & n = \text{even}, \\ a\dfrac{N}{N+2\nu}, & n = \text{odd}. \end{cases} \tag{16}$$

The construction of the mapping (15) is similar to that given in [16].

We also find

$$N = a^\dagger \tilde{a}, \quad N + 1 = \tilde{a} a^\dagger,$$

and

$$\tilde{a} a^{\dagger n}|0\rangle = n a^{\dagger(n-1)}|0\rangle, \quad \langle 0|\tilde{a}^m a^{\dagger n}|0\rangle = n!\delta_{mn}.$$

In the next section we turn to general multi-mode oscillator algebras with permutational invariance.

3 Intermezzo: Multi-mode oscillator algebras with permutation invariance

As we emphasized in [7,8], the analysis of the general multi-mode (deformed) oscillator algebras is more complicated and the three approaches, (6)–(8), become more involved. Here we concentrate on permutation invariant multi-mode oscillator algebras. Invariance on the permutation group simplifies the analysis, but one still has to distinguish between two cases. The first case are permutational invariant algebras with hermitean number operators. These algebras were analyzed in [7,8]. In order to be self-contained, in Sect. 3.1 we repeat the main points of this analysis.

The second case are permutation invariant algebras with non-hermitean number operators, not discussed previously. The analysis of these algebras is presented in Sects. 3.2 and 3.3.

3.1 Permutation invariant algebras with hermitean number operators

Let us consider a system of multi-mode oscillators described by M pairs of creation and annihilation operators $a_i^\dagger$, a_i $(i = 1, 2, \cdots, M)$ hermitian conjugated to each other. We consider operator algebras which possess the well-defined transition number operators N_{ij}, the partial number operators $N_i \equiv N_{ii}$ and the total number operator $N = \sum N_i$. We also demand that the algebras be permutation invariant. In this subsection we suppose that all number operators are *hermitean*, i.e. $N_i^\dagger = N_i$, $N_i^\dagger = N_i$ and $N_{ij}^\dagger = N_{ji}$.

The relations involving the number operators and the operators $a_i^\dagger$, a_i $(i = 1, 2, \cdots, M)$ are

$$\begin{aligned} &[N_{ij}, a_k^\dagger] = \delta_{jk} a_i^\dagger, \quad [N_{ij}, a_k] = -\delta_{ik} a_j, \\ &[N_{ij}, N_{kl}] = \delta_{jk} N_{il} - \delta_{il} N_{kj}, \quad [N_i, a_j] = -\delta_{ij} a_i, \\ &[N_i, a_j^\dagger] = \delta_{ij} a_i^\dagger, \quad [N_i, N_j] = 0, \\ &[N, a_k^\dagger] = a_k^\dagger, \quad [N, a_k] = -a_k. \end{aligned} \tag{17}$$

In the associated Fock-like representation, let $|0\rangle$ denote the vacuum vector. Then, the scalar product is uniquely defined by $\langle 0|0\rangle = 1$, and the vacuum conditions are $a_i|0\rangle = 0$, $a_i a_i^\dagger|0\rangle \neq 0$. A general n-particle state is a linear combination of monomial state vectors $(a_{i_1}^\dagger \cdots a_{i_n}^\dagger|0\rangle)$, $i_1, \cdots, i_n = 1, 2, \cdots, M$. The partial number operators N_i are diagonal on the monomial states $(a_{i_1}^\dagger \cdots a_{i_n}^\dagger|0\rangle)$, with eigenvalues n_i, counting the number of operators $a_i^\dagger$ in the corresponding monomial state:

$$N_i(a_{i_1}^\dagger \underbrace{a_i^\dagger \cdots a_i^\dagger}_{n_i} a_{i_n}^\dagger|0\rangle) = n_i(a_{i_1}^\dagger \underbrace{a_i^\dagger \cdots a_i^\dagger}_{n_i} a_{i_n}^\dagger|0\rangle).$$

In Fock space considered, monomial states with different total number operator N are orthogonal, as well as the states with the same N but different partial number operators N_i. Two states are not orthogonal if they have the same partial number operators $N_1, N_2, \cdots, N_M$.

The general multi-mode oscillator algebra with hermitean number operators and permutation invariance can be described in three ways [7]:

(i) as a normally ordered expansion,
(ii) as a set of relations in the Fock space and,
(iii) using Gram matrix of scalar products in Fock space.

These three approaches are independent but completely equivalent. Moreover, as we demonstrate in the rest of this paper, they can also be applied to the permutation invariant algebras with non-hermitean number operators. Which of the three approaches will be used, depends on the nature of the problem. In the concrete example of the Calogero model (Sect. 4) we are mainly interested in the positivity of the norms in Fock space and in finding the critical points. Therefore, we shall analyze the problem using the Gram matrix.

Now, we shortly describe each of the three approaches.

In the first approach, the operator algebras are defined by a set of relations:

$$\begin{aligned} a_i a_j^\dagger \equiv \Gamma_{ij}(a^\dagger, a) == \delta_{ij} + C_{1,1} a_j^\dagger a_i \\ + \sum_{n=1}^{\infty} \sum_{\pi,\sigma \in S_{n+1}} C_{\pi,\sigma} \sum_{k_1,\cdots,k_n=1}^{M} [\pi(j, k_1, \cdots, k_n)]^\dagger \\ \times [\sigma(i, k_1, \cdots, k_n)], \end{aligned} \tag{18}$$

where the operators a_i are normalized in such a way that the coefficient of the δ_{ij} term is equal to 1. Several comments on the structure of the above expression are in order. Permutation invariance guarantees that the coefficients in the expansion do not depend on concrete indices in normally ordered monomials, but only on certain linearly independent types of permutation invariant terms, schematically displayed above. The existence of the number operators N_i implies that the annihilation and creation operators appearing in a monomial in the normally ordered expansion (18) have to appear in pairs, i.e. monomials are diagonal in the variables $k_1, \cdots, k_n$ (up to permutations) [8]. The symbol $[\sigma(i, k_1, \cdots, k_n)]$ denotes $a_{\sigma(i)} a_{\sigma(k_1)} \cdots a_{\sigma(k_n)} \equiv \sigma(a_i a_{k_1} \cdots a_{k_n})$. Also, $C_{\pi,\sigma} = C^*_{\sigma,\pi}$, owing to the hermiticity of the operator product $a_i a_i^\dagger$. We consider only those relations in (18) that may allow for the norm zero vectors, but do not allow for the state vectors of negative norm in Fock space. The norm zero vectors imply relations between creation (annihilation) operators. Since these relations are consequences of (18), they need not be postulated independently. Also, there is no need to postulate relations $[a_i a_i^\dagger]$ separately, since they can differ from the relations $[a_i a_j^\dagger|_{i=j}]$ only in the unique function $f(N, N_i)$. Finally, notice that, although there are infinitely many terms in the expansion, only finite terms are actually involved when acting on the finite monomial state in Fock space.

In the second approach, in addition to the vacuum relations $a_i|0\rangle = 0$, $a_i a_j^\dagger|0\rangle = \delta_{ij}|0\rangle$, one can define the action of a_i, $i = 1, 2, \cdots, M$, on the monomial states $(a_{i_1}^\dagger \cdots a_{i_n}^\dagger|0\rangle)$ through the relations

$$\begin{aligned} &a_i a_{i_1}^\dagger \cdots a_{i_n}^\dagger|0\rangle \\ &\quad = \sum_{k=1}^{n} \delta_{ii_k} \sum_{\sigma \in S_{n-1}} \phi_\sigma^k [\sigma(i_1, \cdots, \hat{i}_k, \cdots, i_n)^\dagger]|0\rangle, \end{aligned} \tag{19}$$

where $\hat{i}_k$ denotes the omission of the creation operator (with index i_k) in all possible ways. (If the monomial state does not contain $a_i^\dagger$ at all, the RHS of (19) is equal to zero.) The sum is running over all linearly independent monomials, and the ϕ_σ^k are (complex) coefficients. The identity $\phi_{id}^1 = 1$ is implied by normalization in (18). The coefficients ϕ_σ^k can be uniquely determined from $C_{\pi,\sigma}$ and vice versa. If we write the type of monomial state $(a_{i_1}^\dagger \cdots a_{i_n}^\dagger|0\rangle)$ as $1^{\nu_1} 2^{\nu_2} \cdots M^{\nu_M}$, where $\nu_1, \nu_2, \cdots, \nu_M$ are multiplicities satisfying $\nu_i \geq 0$ and $\sum_{i=1}^{M} \nu_i = n$, we see that permutation invariance drastically reduces the number of independent terms in (19), i.e. from M^{n+1} (for the general algebra) to at most

$$\sum_{k=1}^{n} k \frac{n!}{(\nu_1! \cdots \nu_k!)}$$

independent terms (for the permutation invariant algebra).

Example 1. States for $n = 2$: hermitean (N_{ij}, N_i)

$$a_1 (a_1^\dagger)^2|0\rangle, \quad a_1 a_1^\dagger a_2^\dagger|0\rangle, \quad a_2 a_1^\dagger a_2^\dagger|0\rangle.$$

In the third approach, one defines the Gram matrix of scalar products

$$A_{i_n \cdots i_1; j_1 \cdots j_n} = \langle 0 | a_{i_n} \cdots a_{i_1} a^\dagger_{j_1} \cdots a^\dagger_{j_n} | 0 \rangle. \quad (20)$$

For the permutation invariant algebra: that it is permutation invariant means that the matrix element $\langle 0 | a_{i_{\pi(n)}} \cdots a_{i_{\pi(1)}} a^\dagger_{j_{\pi(1)}} \cdots a^\dagger_{j_{\pi(n)}} | 0 \rangle$ does not depend on the permutation $\pi \in S_n$. The rank of the Gram matrix gives the number of linearly independent states in Fock space, which is positive definite if $A \geq 0$, i.e. if all eigenvalues are non-negative. The matrix and its rank depend only on the collection of multiplicities $\{\nu_1, \cdots, \nu_M\}$, which, written in descending order, give rise to a partition of n [8]. The generic matrix (all indices different!) is of the type $n! \times n!$. It can be decomposed in terms of the right regular representation of the permutation group S_n [8]. All other (non-generic) matrices are easily obtained from the generic matrix. Their order is

$$\frac{n!}{\nu_1! \cdots \nu_k!} \times \frac{n!}{\nu_1! \cdots \nu_k!},$$

where $\sum \nu_k = n$.

We mention that the typical permutation invariant algebras with hermitean number operators are parastatistics/interpolation between parastatistics [17] and infinite quon statistics [18]. Finally, we note that there is a simple way to unify a very large class of such permutation invariant algebras as a triple operator algebras [19]:

$$[[a_i, a^\dagger_j]_q, a^\dagger_k] = x\delta_{ij} a^\dagger_k + y\delta_{ik} a^\dagger_j + z\delta_{jk} a^\dagger_i, \quad (x, y, z) \in \mathbf{R}, \quad (21)$$

which can be rewriten as [20]

$$a_i a^\dagger_j = q a^\dagger_j a_i + (1 + xN)\delta_{ij} + yN_{ij} + zN_{ji}.$$

3.2 Permutation invariant algebras with non-hermitean number operators

There exist a large class of operator algebras which possess permutation invariance but for these $N_i \neq N^\dagger_i$ and $N^\dagger_{ij} \neq N_{ji}$ (it still holds $N = N^\dagger$). The most important example is the many-body Calogero model [1,6]. Generally, in these algebras one can have $a_i|0\rangle = 0$, but $a_i a^\dagger_j |0\rangle \neq 0$ for $i \neq j$. As a rule, a non-orthogonal monomial basis appears, i.e. two monomial states with the same total number operator N but different partial number operators N_i are not orthogonal.

The whole algebra can be obtained from one relation, e.g. the one of $a_1 a^\dagger_2$, and by successive application of the permutation $\pi \in S_M$, $a_{\pi(1)} a^\dagger_{\pi(2)} = \pi(a_1 a^\dagger_2)$. The general structure of the normally ordered expansion is

$$a_i a^\dagger_j = c_0 + [a^\dagger_j a_i] + [a^\dagger_j B_{0,1} + B^\dagger_{0,1} a_i] + [B^\dagger_{0,1} B_{0,1}] + [B_{1,1}] + \cdots, \quad (22)$$

where

$$B_{m,n} = \sum_{k=1}^{M} (a^\dagger_k)^m a^n_k, \quad B^\dagger_{m,n} = B_{n,m}.$$

The above expansion displays the S_M-symmetric structure as it contains the S_M-symmetric operators only. Notice that, although the RHS of (22) contains an equal number of $a^\dagger$'s and a's, they are not matched in pairs any longer (cf. (18)). The general structure of the terms in the expansion of $a_i a^\dagger_j$ in (22) (as well as the structure of N_{ij}, N_i and K_{ij}; see below) is

$$a^{\dagger k}_m \mathcal{O} a^l_n, \quad m, n = i, j,$$

where (a_m, a_n) are a_i or a_j and $\mathcal{O}$ is any normally ordered S_M-invariant polynomial in the operators $(a^\dagger, a)$. The indices (i, j) appear explicitly in the expansion and all other indices are contained implicitly in the S_M-invariant operator $\mathcal{O}$. Using this fact, one can uniquely define $[a_i a^\dagger_j|_{i=j}]$ and this should coincide with $[a_i a^\dagger_i]$ up to the unique hermitean function of the operators $(N, N_i, N^\dagger_i, N_{ik}, \cdots)$. which are no longer diagonal on the monomial states (and can change the eigenvalues of N_k for fixed total N).

Going to the Fock-space description (19), we notice that, owing to the non-orthogonality of the monomial states, there appear much more independent terms than in the orthogonal case (see Example 1).

Example 2. States for $n = 2$: non-hermitean (N_{ij}, N_i),

$$a_1 (a^\dagger_1)^2 |0\rangle, \quad a_2 (a^\dagger_1)^2 |0\rangle, \quad a_1 a^\dagger_1 a^\dagger_2 |0\rangle, \quad a_2 a^\dagger_1 a^\dagger_2 |0\rangle, \quad a_3 a^\dagger_1 a^\dagger_2 |0\rangle.$$

Generally, there are at most

$$\sum_{k=1}^{n} (k+1) \frac{n!}{\nu_1! \cdots \nu_k!}, \quad \sum_{i=1}^{k} \nu_i = n$$

independent terms.

The Gram matrices $A_{i_n \cdots i_1; j_1 \cdots j_n}$ are hermitean, of the type M^n and we require all eigenvalues to be non-negative. The matrix elements of the particular Gram matrix are related by permutation symmetry. In the next section we study the structure of these matrices in more detail.

3.3 Operators N_{ij}, N_i and K_{ij}

The transition number operators N_{ij} can be expanded into an infinite (normally ordered) series in creation and annihilation operators. However, only finitely many terms are involved when N_{ij} acts on a finite monomial state in Fock space.

We find that the general structure of N_{ij} is (cf. (22))

$$N_{ij} = [a^\dagger_i a_j] + [a^\dagger_i B_{0,1} + B^\dagger_{0,1} a_j] + [B^\dagger_{0,1} B_{0,1}] + [B_{1,1}] + \cdots \quad (23)$$

The partial number operators N_i are obtained from the above expression as $N_i = N_{ii}$. Notice that $N_{ij}^\dagger \neq N_{ji}$.

The structure of the total number operator $N = \sum N_i$ is

$$\begin{aligned} N &= [B_{0,1}^\dagger B_{0,1}] + [B_{1,1}] + [B_{0,2}^\dagger B_{0,2}] \\ &\quad + [B_{0,2}^\dagger B_{0,1}^2] + [B_{0,1}^{\dagger 2} B_{0,2}] + [B_{0,1}^{\dagger 2} B_{0,1}^2] \\ &\quad + [B_{2,1} B_{0,1}] + [B_{0,1}^\dagger B_{1,2}] + [B_{2,2}] + \cdots \end{aligned} \quad (24)$$

As we are interested only in the S_M-symmetric structure, the coefficients in the above expressions have been omitted. Notice that $N^\dagger = N$.

The exchange operators K_{ij}, $i,j = 1,2,\cdots,M$, generate the symmetric group S_M. They are defined as follows:

$$\begin{aligned} K_{ij} = K_{ji}, \quad (K_{ij})^2 = 1, \quad K_{ij}^\dagger = K_{ij}, \\ K_{ij}K_{jl} = K_{jl}K_{il} = K_{il}K_{ij}, \quad i \neq j,\ i \neq l,\ j \neq l. \end{aligned} \quad (25)$$

The representation of the symmetric group S_M exists on every deformed algebra of a_i and $a_i^\dagger$ $(i = 1, \cdots, M)$ in the following sense:

$$\begin{aligned} K_{ij}a_j = a_i K_{ij}, \quad K_{ij}a_k = a_k K_{ij}, \\ K_{ij}a_j^\dagger = a_i^\dagger K_{ij}, \quad K_{ij}a_k^\dagger = a_k^\dagger K_{ij}, \end{aligned} \quad (26)$$

for $k \neq i$ and $k \neq j$.

The vacuum condition is $K_{ij}|0\rangle = \pm|0\rangle$, and we choose $K_{ij}|0\rangle = +|0\rangle$.

Remark 2. Note that the following change of the definitions in (26):

$$K_{ij}a_j = -a_i K_{ij}, \quad K_{ij}a_k = +a_k K_{ij},$$

(and similarly for $a_i^\dagger$), directly leads to contradiction since, from (25) and (26), we obtain two apparently different results:

$$\begin{aligned} K_{ij}K_{jk}a_k = a_i K_{ij}K_{jk}, \\ K_{jk}K_{ki}a_k = -a_i K_{jk}K_{ki}. \end{aligned}$$

The important fact is that if the algebra of the oscillators is permutation invariant, then the K_{ij} operators can be expressed similarly as N, N_i, N_{ij}, namely as an infinite series expansion in the creation and annihilation operators. If the algebra is not permutation invariant, such a representation of exchange operators may not exist. We point out the difference between the operators N_{ij} and K_{ij}. The exchange operators K_{ij} act "globally" (and simultaneously) on the right on any monomial in $a^\dagger$ and a, interchanging indices i and j and keeping all other indices at rest. Transition number operators N_{ij} act "locally", turning only one $a_j^\dagger$ into $a_i^\dagger$, at the same place where $a_j^\dagger$ is sitting. The action of N_{ij} can be repeated at most n_j times, where n_j counts the number of $a_j^\dagger$'s in the monomial. If there is only one $a_j^\dagger$, then we have

$$N_{ij}(\cdots a_j^\dagger \cdots)|0\rangle = (\cdots a_i^\dagger \cdots)|0\rangle.$$

Let $[a_i^\dagger, a_j^\dagger] = 0$ for $i \neq j$. We symbolically denote the eigenstate of N_i as $(\cdots a_i^{\dagger n_i} \cdots a_j^{\dagger n_j} \cdots)|0\rangle$. Then

$$\begin{aligned} & N_i(\cdots a_i^{\dagger n_i} \cdots a_j^{\dagger n_j} \cdots)|0\rangle \\ &\quad = n_i(\cdots a_i^{\dagger n_i} \cdots a_j^{\dagger n_j} \cdots)|0\rangle, \\ & N_j(\cdots a_i^{\dagger n_i} \cdots a_j^{\dagger n_j} \cdots)|0\rangle \\ &\quad = n_j(\cdots a_i^{\dagger n_i} \cdots a_j^{\dagger n_j} \cdots)|0\rangle, \\ & N_{ij}(\cdots a_i^{\dagger n_i} \cdots a_j^{\dagger n_j} \cdots)|0\rangle \\ &\quad = n_j(\cdots a_i^{\dagger n_i+1} \cdots a_j^{\dagger n_j-1} \cdots)|0\rangle, \\ & N_{ij}^{n_j}(\cdots a_i^{\dagger n_i} \cdots a_j^{\dagger n_j} \cdots)|0\rangle \\ &\quad = n_j!(\cdots a_i^{\dagger n_i+n_j} \cdots a_j^{\dagger 0} \cdots)|0\rangle, \\ & N_{ji}^{n_i} N_{ij}^{n_j}(\cdots a_i^{\dagger n_i} \cdots a_j^{\dagger n_j} \cdots)|0\rangle \\ &\quad = (n_i+n_j)!(\cdots a_i^{\dagger n_j} \cdots a_j^{\dagger n_i} \cdots)|0\rangle. \end{aligned}$$

We also obtain

$$\begin{aligned} K_{ij} &= \frac{1}{(n_i+n_j)!}(N_{ji})^{n_i}(N_{ij})^{n_j} \\ &= \frac{1}{(n_i+n_j)!}(N_{ij})^{n_j}(N_{ji})^{n_i}, \end{aligned} \quad (27)$$

or alternatively

$$K_{ij} = \begin{cases} \left(N_{ij}\dfrac{1}{N_j}\right)^{n_j-n_i} & \text{if } n_i < n_j, \\ 1 & \text{if } n_i = n_j, \\ \left(N_{ji}\dfrac{1}{N_i}\right)^{n_i-n_j} & \text{if } n_i > n_j, \end{cases}$$

where n_i and n_j again denote, respectively, the number of $a_i^\dagger$ and $a_j^\dagger$ in the monomial $(\cdots a_i^\dagger \cdots a_j^\dagger)|0\rangle$. If $[a_i^\dagger, a_j^\dagger] \neq 0$ for $i \neq j$, there is generally no such simple relation between K_{ij} and N_{ij}.

Below we give two examples of K_{ij} operators for permutation invariant algebras with hermitean number operators. An example of K_{ij} operators for permutation invariant algebra with non-hermitean number operators is given in the next section.

Example 3. Heisenberg algebra of Bose oscillators b_i, $b_i^\dagger$, $i = 1, \cdots, M$.

Algebra:

$$[b_i, b_j^\dagger] = \delta_{ij}, \quad [b_i^\dagger, b_j^\dagger] = [b_i, b_j] = 0.$$

Number operators:

$$N_{ij} = b_i^\dagger b_j, \quad N_i = b_i^\dagger b_i, \quad N = \sum_{i=1}^M b_i^\dagger b_i.$$

Exchange operators:

$$K_{ij} =: e^{-(b_i^\dagger - b_j^\dagger)(b_i - b_j)} := \sum_{k=0}^\infty \frac{(-)^k}{k!}(b_i^\dagger - b_j^\dagger)^k (b_i - b_j)^k.$$

Example 4. Clifford algebra of Fermi oscillators f_i, $f_i^\dagger$, $i = 1, \cdots, M$.

Algebra:

$$\{f_i, f_j^\dagger\} = \delta_{ij}, \quad \{f_i^\dagger, f_j^\dagger\} = \{f_i, f_j\} = 0.$$

Number operators:

$$N_{ij} = f_i^\dagger f_j, \quad N_i = f_i^\dagger f_i, \quad N = \sum_{i=1}^{M} f_i^\dagger f_i.$$

Exchange operators:

$$K_{ij} =: e^{-(f_i^\dagger - f_j^\dagger)(f_i - f_j)} := 1 - (f_i^\dagger - f_j^\dagger)(f_i - f_j).$$

4 Application: The M-body Calogero model

The results of a general analysis of permutation invariant multi-mode oscillator algebras with non-hermitean number operators (Sects. 3.2 and 3.3) will be now applied to the M-body Calogero model. We also generalize several concepts (e.g. the infinite series expansion of $a_i a_j^\dagger$, N, N_i, N_{ij}, K_{ij} and dual algebra) introduced in Sect. 2. We find, as a particularly interesting result, the structure and the eigensystem of the Gram matrices. The analysis of the Gram matrices enables us to locate the universal critical point of the M-body Calogero model at $\nu = -1/M$.

4.1 The M-body Calogero model and the multi-mode oscillator algebras

The M-body Calogero model, describing M identical bosons on the line, is defined by the following Hamiltonian [1]:

$$H = -\frac{1}{2}\sum_{i=1}^{M} \partial_i^2 + \frac{1}{2}\sum_{i=1}^{M} x_i^2 + \frac{\nu(\nu-1)}{2}\sum_{i\neq j}^{M} \frac{1}{(x_i - x_j)^2}. \quad (28)$$

For simplicity, we have set $\hbar$, the mass of the particles and the frequency of harmonic oscillators equal to one. The dimensionless constant ν is the coupling constant (and/or the statistical parameter) and M is the number of particles. The Hamiltonian (28) can be factorized by the the creation and annihilation operators of the S_M-extended Heisenberg algebra [6].

Let us introduce the following analogs of the creation and annihilation operators [6]:

$$a_i^\dagger = \frac{1}{\sqrt{2}}(-D_i + x_i), \quad a_i = \frac{1}{\sqrt{2}}(D_i + x_i),$$

where

$$D_i = \partial_i + \nu \sum_{j, j\neq i}^{M} \frac{1}{x_i - x_j}(1 - K_{ij})$$

are Dunkl derivatives and K_{ij} are exchange operators (see (25) and (26)) generating the symmetric group S_M. One can easily check that the commutators of the creation and annihilation operators are

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0,$$
$$[a_i, a_j^\dagger] = A_{ij} = \left(1 + \nu \sum_{k=1}^{M} K_{ik}\right)\delta_{ij} - \nu K_{ij}. \quad (29)$$

The action of K_{ij} on $a_i^\dagger$ and a_i is given in (26).

Remark 3. The following definitions are also consistent:

$$K_{ij} f_j = -f_i K_{ij},$$
$$K_{ij} f_k = -f_k K_{ij},$$

and one can study the algebra defined by the anticommutator $\{a_i, a_j^\dagger\}$:

$$\{a_i, a_j^\dagger\} = A_{ij} = \left(1 + \nu \sum_{k=1}^{N} K_{ik}\right)\delta_{ij} - \nu K_{ij}, \quad \forall i, j.$$

Note that $\{a_i, a_j\} \neq 0$, if $\nu \neq 0$. Moreover, one can show that the general form of the algebra, namely $[a_i, a_j^\dagger]_q = A_{ij}(\nu)$, $|q| \leq 1$, with the condition $K_{ij}a_j = a_i K_{ij}$, (26), is equivalent to the algebra $[a_i, a_j^\dagger]_q = A_{ij}(-\nu)$, with the conditions $K_{ij}a_j = -a_i K_{ij}$ and $K_{ij}a_k = -a_k K_{ij}$.

After performing a similarity transformation on the Hamiltonian H $(\prod_{i<j}^{M} |x_i - x_j|^{-\nu}) H (\prod_{i<j}^{M} |x_i - x_j|^{\nu})$ we obtain the reduced Hamiltonian H' which, when restricted to the space of symmetric functions, takes the following simple form:

$$H' = \frac{1}{2}\sum_{i=1}^{M}\{a_i, a_i^\dagger\} = \sum_{i=1}^{M} a_i^\dagger a_i + E_0,$$
$$[H', a_i] = -a_i, \quad [H', a_i^\dagger] = a_i^\dagger, \quad [H', \{a_i, a_j^\dagger\}] = 0. \quad (30)$$

Notice that one can define the general Hamiltonian $\bar{H}$ by

$$\bar{H} = \frac{1}{2}\sum_{i=1}^{M}\{a_i, a_i^\dagger\} = N + E_0,$$

which is not restricted to the symmetric states only, i.e. it acts on the states in the whole Fock space. In the space of symmetric states, it coincides with H', $H' = \bar{H}$. The ground-state energy is $E_0 = (M/2)(1 + \nu(M-1))$. The Fock-space representation is defined by $a_i|0\rangle = 0$, $\forall i$ and $K_{ij}|0\rangle = \epsilon|0\rangle$, $\forall(i,j)$. As $\epsilon^2 = 1$, we fix ϵ to $+1$. The Fock space is spanned by the monomials $(a_1^{\dagger n_1} \cdots a_M^{\dagger n_M}|0\rangle)$. In the following we analyze the full Fock space of states since

(i) we want to apply ideas from Sect. 3 to the S_M-extended Heisenberg algebra (29) and
(ii) we want to obtain the positivity of the physical states as a consequence of the positivity of states in the complete Fock space.

Physical states are symmetric (antisymmetric) states for the bosonic (fermionic) systems. The algebraic analysis of the physical space of symmetric states for a two- and a three-body Calogero model (28) is given in [21]. A general approach to the algebra of observables and dynamical symmetry algebra for the M-body Calogero model was proposed in [22].

We point out that the algebra (29) can be defined in a new way, without exchange operators K_{ij}. The construction relies on the generalization of the triple operator algebras (21) [19,20]. The only difference is that now the number operators N_i and N_{ij} are not *hermitean*.

Eliminating $\nu K_{ij} = [a_i, a_j^\dagger]$ for $i \neq j$, we find

$$\begin{aligned} [a_i, B_{0,1}^\dagger] &= 1, \quad \forall i, \\ a_i[a_i, a_j^\dagger] &= [a_i, a_j^\dagger]a_j, \quad \forall (i,j) i \neq j, \\ a_i[a_j, a_i^\dagger] &= [a_j, a_i^\dagger]a_j, \quad \forall (i,j), i \neq j, \\ a_k[a_i, a_j^\dagger] &= [a_i, a_j^\dagger]a_k, \quad \forall (i,j,k) i \neq j \neq k \neq i. \end{aligned} \tag{31}$$

(It is understood that the hermitean counterparts of these relations also hold.) Notice that the single-mode algebra (3) is a true triple operator algebra (21) since it can be rewritten as $[\{a, a^\dagger\}, a^\dagger] = 2a^\dagger$ and $[\{a, a^\dagger\}, a] = -2a$.

The algebra (31) is still a permutation invariant algebra, but the indices on the LHS and RHS are not the same (cf. the second and third relations in (31)). The last relation in (31) can be written as

$$\begin{aligned} [a_k, [a_i, a_j^\dagger]] &= 0, \quad \forall (i,j,k), i \neq j \neq k \neq i, \\ [a_i, a_j^\dagger] &= [a_j, a_i^\dagger], \quad \forall (i,j). \end{aligned}$$

For the triple operator algebras, the Fock-space representation is defined by the two generalized vacuum conditions (one can notice the similarity with Green's parastatistics [17]):

$$\begin{aligned} a_i|0\rangle &= 0, \quad \forall i, \\ a_i a_j^\dagger|0\rangle &= -\nu|0\rangle, \quad \forall (i,j), i \neq j. \end{aligned} \tag{32}$$

Under these conditions, the Fock representation is uniquely determined and is equivalent to the first construction, (29). However, this algebra does not depend on ν and K_{ij}, i.e. in the formulation of the triple operator algebras, the interaction parameter ν enters only through the vacuum condition (32).

We also obtain the consistency conditions in the Fock representation, namely

$$\begin{aligned} ([a_i, a_j^\dagger])^2 = \nu^2, &\quad [a_k, ([a_i, a_j^\dagger])^2] = 0, \\ [a_i, a_j^\dagger][a_j, a_k^\dagger] = [a_j, a_k^\dagger][a_i, a_k^\dagger] &= [a_i, a_k^\dagger][a_i, a_j^\dagger]. \end{aligned} \tag{33}$$

There is a simple generalization of the triple operator algebra to include fermions (i.e. anticommutators) and quons (i.e. q-commutators).

4.2 Gram matrices for the Calogero model

In the rest of this section, we discuss the Calogero model in the framework of the three approaches proposed in Sect. 3. Special attention is given to the Gram matrix approach.

In the first approach we express $a_i a_j^\dagger$ as a normally ordered expansion:

$$\begin{aligned} a_i a_j^\dagger &= -\nu K_{ij} + a_j^\dagger a_i, \quad \forall (i,j), i \neq j, \\ a_i a_i^\dagger &= 1 + a_i^\dagger a_i + \nu \sum_{l, l \neq i} K_{il}. \end{aligned} \tag{34}$$

This is obviously a permutation invariant algebra. We can write it in the form of (22) if we know the expansion of K_{ij} in terms of a_i and $a_j^\dagger$ (and vice versa). Later we shall give such a construction.

In the second approach we have to know the action of a_i on the monomial states $(a_1^{\dagger n_1} \cdots a_M^{\dagger n_M}|0\rangle)$ in the Fock space.

For the one-particle states $(a_i^\dagger|0\rangle)$, using $a_i|0\rangle = 0$, $\forall i$, we find

$$a_i a_j^\dagger|0\rangle = \begin{cases} -\nu|0\rangle, & i \neq j, \\ (1 + (M-1))|0\rangle, & \forall i = j. \end{cases}$$

For the two-particle states $(a_i^\dagger a_j^\dagger|0\rangle$) we find

$$a_i a_{j_1}^\dagger a_{j_2}^\dagger|0\rangle = [A_{ij_1} a_{j_2}^\dagger + a_{j_1}^\dagger A_{ij_2}]|0\rangle.$$

There are only four types of relations, owing to $[a_{j_1}^\dagger, a_{j_2}^\dagger] = 0$:

$$\begin{aligned} a_1(a_1^\dagger)^2|0\rangle &= (\nu(M-2)+2)a_1^\dagger|0\rangle + \nu B_{0,1}^\dagger|0\rangle, \\ a_1(a_2^\dagger)^2|0\rangle &= -\nu(a_1^\dagger + a_2^\dagger)|0\rangle, \\ a_1 a_1^\dagger a_2^\dagger|0\rangle &= (\nu(M-2)+1)a_2^\dagger|0\rangle, \\ a_1 a_2^\dagger a_3^\dagger|0\rangle &= -\nu(a_3^\dagger + a_2^\dagger)|0\rangle. \end{aligned}$$

For the three-particle states $(a_i^\dagger a_j^\dagger a_k^\dagger|0\rangle)$ we find

$$a_i a_{j_1}^\dagger a_{j_2}^\dagger a_{j_3}^\dagger|0\rangle = [A_{ij_1} a_{j_2}^\dagger a_{j_3}^\dagger + a_{j_1}^\dagger A_{ij_2} a_{j_3}^\dagger + a_{j_1}^\dagger a_{j_2}^\dagger A_{ij_3}]|0\rangle.$$

There are seven types of relations:

$$\begin{aligned} a_1(a_1^\dagger)^3|0\rangle &= (\nu(M-3)+3)a_1^{\dagger 2}|0\rangle \\ &\quad + \nu(a_1^\dagger B_{0,1}^\dagger + B_{0,2}^\dagger)|0\rangle, \\ a_1(a_2^\dagger)^3|0\rangle &= -\nu(a_1^\dagger a_2^\dagger + a_1^{\dagger 2} + a_2^{\dagger 2})|0\rangle, \\ a_1(a_1^\dagger)^2 a_2^\dagger|0\rangle &= (\nu(M-2)+2)a_1^\dagger a_2^\dagger|0\rangle \\ &\quad - \nu a_2^{\dagger 2}|0\rangle + \nu a_2^\dagger B_{0,1}^\dagger|0\rangle, \\ a_1 a_1^\dagger (a_2^\dagger)^2|0\rangle &= (\nu(M-2)+1)a_2^{\dagger 2}|0\rangle - \nu a_1^\dagger a_2^\dagger|0\rangle, \\ a_1 a_1^\dagger a_2^\dagger a_3^\dagger|0\rangle &= (\nu(M-3)+1)a_2^\dagger a_3^\dagger|0\rangle, \\ a_1(a_2^\dagger)^2 a_3^\dagger|0\rangle &= -\nu(a_2^\dagger a_3^\dagger + a_2^{\dagger 2} + a_1^\dagger a_3^\dagger)|0\rangle, \\ a_1 a_2^\dagger a_3^\dagger a_4^\dagger|0\rangle &= -\nu(a_2^\dagger a_3^\dagger + a_2^\dagger a_4^\dagger + a_3^\dagger a_4^\dagger)|0\rangle. \end{aligned}$$

It is a simple task to generalize these equations to an arbitrary multi-particle state, i.e. $((a_1^\dagger)^{n_1}\cdots(a_M^\dagger)^{n_M})|0\rangle)$:

$$\begin{aligned}
&a_1(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}\cdots(a_M^\dagger)^{n_M})|0\rangle \equiv a_1|n_1;n_2;\cdots n_M\rangle\\
&= n_1|n_1-1;n_2;n_3;\cdots n_M\rangle\\
&+\nu\,\mathrm{sgn}(n_1-n_2)\sum_{k=1}^{|n_1-n_2|}|\min(n_1,n_2)\\
&\quad +k-1;\max(n_1,n_2)-k;n_3;\cdots n_M\rangle\\
&+\nu\,\mathrm{sgn}(n_1-n_3)\sum_{k=1}^{|n_1-n_3|}|\min(n_1,n_3)\\
&\quad +k-1;n_2;\max(n_1,n_3)-k;\cdots n_M\rangle+\cdots\\
&+\nu\,\mathrm{sgn}(n_1-n_M)\sum_{k=1}^{|n_1-n_M|}|\min(n_1,n_M)\\
&\quad +k-1;n_2;\cdots;\max(n_1,n_M)-k\rangle. \qquad (35)
\end{aligned}$$

We use these formulas in the constructing of the Gram matrices for different M (the third approach).

Now we easily obtain the structure of the matrix elements of the Gram matrices $\langle 0|a_{i_n}\cdots a_{i_1}a_{j_1}^\dagger\cdots a_{j_n}^\dagger|0\rangle$. We explicitly give several examples (for $M=2,3$) in the appendix and below we discuss eigenvalues and eigenvectors of Gram matrices corresponding to one- and two-particle states for any M.

(1) One-particle states ($a_i^\dagger|0\rangle$, $i=1,2,\cdots,M$): The matrix of one-particle states is of order M and has only two distinct entries: $-\nu$ and $1+\nu(M-1)$. Its eigenvalues and typical eigenvectors are written in Table 1.

The positivity condition implies that all eigenvalues should be positive, meaning $1+M\nu>0$ or $\nu>-1/M$.

(2) Two-particle states ($a_i^\dagger a_j^\dagger|0\rangle$, $(i,j)=1,2,\cdots M$): The matrix of two-particle states is of order M^2 and has four distinct entries of the form

$$\begin{aligned}
\langle 0|a_1^2(a_1^\dagger)^2|0\rangle &= a,\\
\langle 0|a_1^2(a_2^\dagger)^2|0\rangle &= \langle 0|a_1^2a_2^\dagger a_3^\dagger|0\rangle = \langle 0|a_1a_2a_3^\dagger a_4^\dagger|0\rangle = b,\\
\langle 0|a_1^2a_1^\dagger a_2^\dagger|0\rangle &= \langle 0|a_1a_2a_1^\dagger a_3^\dagger|0\rangle = c,\\
\langle 0|a_1a_2a_1^\dagger a_2^\dagger|0\rangle &= d,
\end{aligned}$$

where $a=[1+\nu(M-1)][2+\nu(M-1)]-\nu^2(N-1)$, $b=-\nu-\nu^2(M-2)$, $c=2\nu^2$ and $d=[1+\nu(M-1)][2+\nu(M-2)]$.

Its eigenvalues and typical eigenvectors are written in Table 2.

Here, $B_{0,1}=\sum_i a_i$ and $B_{0,2}=\sum_i a_i^2$. Note that null-eigenstates are identically equal to zero owing to the commutation relation $[a_i^\dagger,a_j^\dagger]\equiv 0$, which is satisfied by $\forall(i,j)$.

The positivity condition implies again that all non-zero eigenvalues are positive, which is satisfied if $1+M\nu>0$, i.e. $\nu>-1/M$.

One can show that the same condition for the positivity of the eigenvalues for three and more particle states also holds. There is a universal critical point, $\nu=-1/M$, at which all matrix elements of an arbitrary multi-state

Table 1.

Eigenvalue	Degeneracy	Eigenvector	Comments
1	1	$B_{0,1}^\dagger|0\rangle$	
$1+M\nu$	$M-1$	$(a_1^\dagger-a_i^\dagger)|0\rangle$	$i\neq 1$

Gram matrix are equal to $(k!/M^k)$, where k denotes a k-particle state. This can be proved by induction and here we sketch the proof.

The generic Gram matrix is of type $(M^k\times M^k)$. At the critical point $\nu=-1/M$ we find

$$\begin{aligned}
A_{ij} &\equiv \langle 0|a_ia_j^\dagger|0\rangle = \frac{1}{M},\\
A_{i_2i_1;j_1j_2} &\equiv \langle 0|a_{i_2}a_{i_1}a_{j_1}^\dagger a_{j_2}^\dagger|0\rangle\\
&= A_{i_1j_1}\langle 0|a_{i_2}a_{j_2}^\dagger|0\rangle + A_{i_1j_2}\langle 0|a_{i_2}a_{j_1}^\dagger|0\rangle\\
&= \frac{2}{M^2},\\
&\vdots\\
A_{i_k\cdots i_1;j_1\cdots j_k} &\equiv \langle 0|a_{i_k}\cdots a_{i_1}a_{j_1}^\dagger\cdots a_{j_k}^\dagger|0\rangle\\
&= A_{i_1j_1}\langle 0|a_{i_k}\cdots a_{i_2}a_{j_2}^\dagger\cdots a_{j_k}^\dagger|0\rangle\\
&+ A_{i_1j_2}\langle 0|a_{i_k}\cdots a_{i_2}a_{j_1}^\dagger a_{j_3}^\dagger\cdots a_{j_k}^\dagger|0\rangle+\cdots\\
&+ A_{i_1j_k}\langle 0|a_{i_k}\cdots a_{i_2}a_{j_1}^\dagger\cdots a_{j_{k-1}}^\dagger|0\rangle\\
&= k\frac{1}{M}\langle 0|a_{i_k}\cdots a_{i_2}a_{j_2}^\dagger\cdots a_{j_k}^\dagger|0\rangle = \frac{k!}{M^k}.
\end{aligned}$$

The rank of this matrix is one. There is only one state, namely $B_{0,1}^{\dagger k}|0\rangle$, which corresponds to the center of mass and has positive norm. The corresponding eigenvalue is $(k!/M^k)M^k=k!$. All other eigenstates are null-states with zero norm. Diagonal matrix elements are larger than $(k!/M^k)$ if $\nu>-1/M$ and the positivity conditions for eigenvalues are satisfied.

Remark 4. The critical point is universal in the sense that all algebras of the form $[a_i,a_j^\dagger]_q=A_{ij}(\nu)$, $|q|\leq 1$ have the same critical point $\nu=-1/M$. The case $q=-1$, for which the algebra takes a fermionic form, is of special interest:

$$\{f_i,f_j^\dagger\}=A_{ij},\quad \{F,F^\dagger\}=M,$$
$$\{f_i,F^\dagger\}=1,\quad F^2=F^{\dagger 2}=0.$$

Here, $F=\sum f_i$. It follows that the one-particle Gram matrix for $q=-1$ is the same as for $q=1$:

$$\begin{pmatrix}
1+\nu(M-1) & -\nu & \cdots & -\nu\\
-\nu & 1+\nu(M-1) & \cdots & -\nu\\
\cdots & \cdots & \cdots & \cdots\\
-\nu & -\nu & \cdots & 1+\nu(M-1)
\end{pmatrix}.$$

The matrices for two- and many-particle cases depend on q and will be treated in separate paper. It appears that $\nu=-1/M$ could be interpreted as a physically interesting point [23]. At this point, the Fock space reduces to the Fock space of a single harmonic oscillator, corresponding to the centre-of-mass coordinate.

Table 2.

Eigenvalue	Degeneracy	Eigenvector	Comments
0	$M(M-1)/2$	$[a_i^\dagger, a_j^\dagger]\lvert 0\rangle$	$\forall(i,j), i \neq j$
$2(1+M\nu)$	M	$(\{a_i^\dagger, B_{0,1}^\dagger\} - 2B_{0,2}^\dagger)\lvert 0\rangle$	$\forall i, M \geq 2$
$(1+M\nu)(2+M\nu)$	$M-1$	$(\{(a_i^\dagger - a_1^\dagger), B_{0,1}^\dagger\} - M(a_i^{\dagger 2} - a_1^{\dagger 2}))\lvert 0\rangle$	$i \neq 1; M \geq 3$
$2(1+M\nu)(1+\nu(M-1))$	$M(M-3)/2$	$\{(a_i^\dagger - a_j^\dagger), (a_k^\dagger - a_l^\dagger)\}\lvert 0\rangle$	$(i \neq j \neq k \neq l)$

4.3 Operators N_{ij}, N_i, N and K_{ij}

Now, we proceed to the construction of N_{ij}, N_i, N and K_{ij} operators. This construction can be performed for any M but, for simplicity, we present the results for the first non-trivial case $M = 3$. All these constructions exist only if the positivity condition, $\nu > -1/M$, is satisfied. For $M = 3$, $\nu > -1/3$. The construction starts with expanding the corresponding operator in a series in a_i and $a_i^\dagger$; for example (indices are omitted for brevity),

$$K_{ij} = c_0 + \sum c_1 a^\dagger a + \sum c_2 a^\dagger a^\dagger a a + \cdots \tag{36}$$

Using the definitions (25) and (26), we act with (36) on the vacuum (which gives $c_0 = 1$), then on the one-particle state, the two-particle state, etc. In this way, we obtain linear recursive relations which are easily solved. The result for K_{12} and $M = 3$ is

$$\begin{aligned} K_{12} = 1 &- \frac{1}{(1+3\nu)} b_{12}^\dagger b_{12} + \frac{1}{2(1+3\nu)^2} b_{12}^{\dagger 2} b_{12}^2 \\ &- \frac{\nu}{2(1+3\nu)^2(2+3\nu)} b_{12}^\dagger b_{123}^\dagger b_{12} b_{123} + \cdots, \end{aligned} \tag{37}$$

where $b_{12} = a_1 - a_2$ and $b_{123} = a_1 + a_2 - 2a_3$. One gets K_{13} and K_{23} from K_{12} using permutation invariance. Knowing K_{ij}, one can find a normally ordered expansion $a_i a_j^\dagger$.

Similarly, one finds ($M = 3$)

$$\begin{aligned} N_1 = &\frac{1}{(1+3\nu)} a_1^\dagger a_1 + \frac{\nu}{(1+3\nu)} a_1^\dagger B_{0,1} \\ &- \frac{\nu}{4(1+3\nu)(2+3\nu)} a_1^\dagger b_{231}^\dagger b_{23}^2 \\ &- \frac{\nu(1+\nu)}{4(1+3\nu)^2(2+3\nu)} a_1^\dagger b_{231}^\dagger b_{231}^2 \\ &- \frac{\nu}{2(1+3\nu)^2(2+3\nu)} a_1^\dagger b_{23}^\dagger b_{23} b_{231} + \cdots \end{aligned} \tag{38}$$

Here, $b_{23} = a_2 - a_3$ and $b_{231} = a_2 + a_3 - 2a_1$. N_{12} is easy to obtain from the above formula. Note that $N_1^\dagger \neq N_1$. Similarly, $N_{12}^\dagger \neq N_{21}$. However, the total number operator N is hermitean, $N^\dagger = N$, and for $M = 3$:

$$\begin{aligned} N = &\frac{1}{(1+3\nu)} \sum_{i=1}^{3} a_i^\dagger a_i + \frac{\nu}{(1+3\nu)} \left(\sum_{i=1}^{3} a_i^\dagger\right)\left(\sum_{i=1}^{3} a_i\right) \\ &+ \frac{\nu}{(1+3\nu)^2(2+3\nu)} \sum_{i<j=1}^{3} (a_i^\dagger - a_j^\dagger)^2 (a_i - a_j)^2 \\ &+ \frac{2\nu^2}{(1+3\nu)^2(2+3\nu)} \left[\sum_{i=1}^{3} a_i^{\dagger 2} - \sum_{i<j=1}^{3} a_i^\dagger a_j^\dagger\right] \\ &\times \left[\sum_{i=1}^{3} a_i - \sum_{i<j=1}^{3} a_i a_j\right] \\ \equiv &\frac{1}{(1+3\nu)} B_{1,1} + \frac{\nu}{(1+3\nu)} B_{0,1}^\dagger B_{0,1} \\ &+ \frac{\nu}{(1+3\nu)^2(2+3\nu)} \\ &\times \left\{2\nu \left[\frac{3}{2} B_{0,2}^\dagger - \frac{1}{2} B_{0,1}^{\dagger 2}\right]\left[\frac{3}{2} B_{0,2} - \frac{1}{2} B_{0,1}^2\right]\right. \\ &+ 3B_{2,2} + B_{0,2}^\dagger B_{0,2} - 2(B_{2,1}B_{0,1} + \text{h.c.}) \\ &\left.+ 2\sum_{i=1}^{3} a_i^\dagger B_{1,1} a_i \right\}. \end{aligned} \tag{39}$$

The result is consistent with the general expression $\bar{H} - E_0 = (1/2)\sum_i \{a_i, a_i^\dagger\} - E_0 = N$. In the limit $\nu \to 0$, we reproduce the standard result $N = \sum_i a_i^\dagger a_i$. Although the above expressions seem to be divergent at the critical point, it appears that for $\nu = -1/3$, the degrees of freedom, the Fock space and the related algebra are substantially reduced [23] and the above expansions are completely regular, giving $N = (1/3) B_{0,1}^\dagger B_{0,1}$ at $\nu = -1/3$.

4.4 Dual operators $\tilde{a}_i$ and dual algebra $\tilde{\mathcal{A}}$

Owing to the fact that $[a_i, a_j] = 0$ and $[a_i^\dagger, a_j^\dagger] = 0$, $\forall(i,j)$, we can define the operators $\tilde{a}_i$, $i = 1, 2, \cdots, M$, $\nu > -1/M$, such that

$$\begin{aligned} \tilde{a}_i (a_{i_1}^\dagger \cdots a_{i_m}^\dagger \lvert 0\rangle) &= \sum_{\alpha=1}^{m} \delta_{i i_\alpha} a_{i_1}^\dagger \cdots \hat{a}_{i_\alpha}^\dagger \cdots a_{i_m}^\dagger \lvert 0\rangle, \\ \tilde{a}_i \lvert 0\rangle &= 0, \end{aligned} \tag{40}$$

where the hat denotes omission of the corresponding operator.

The sum on the RHS contains m_i terms. We find that

$$\begin{aligned} &\tilde{a}_i (a_{i_1}^{\dagger m_1} \cdots a_i^{\dagger m_i} \cdots a_{i_M}^{\dagger m_M} \lvert 0\rangle) \\ &\quad = m_i (a_{i_1}^{\dagger m_1} \cdots a_i^{\dagger m_i - 1} \cdots a_{i_M}^{\dagger m_M} \lvert 0\rangle) \end{aligned}$$

and

$$[\tilde{a}_i, a_j^\dagger] = \delta_{ij}, \quad [\tilde{a}_i, \tilde{a}_j] = 0, \quad \forall(i,j). \tag{41}$$

These relations are satisfied on all monomial states in Fock space. If we define a dual Fock space, spanned by monomials $(\langle 0|\tilde{a}_{i_1}\cdots\tilde{a}_{i_M})$, we obtain the following relation, as a consequence of (41):

$$\langle 0|\tilde{a}_{i_1}^{\tilde{m}_1}\cdots\tilde{a}_{i_M}^{\tilde{m}_M}a_{i_1}^{\dagger m_1}\cdots a_{i_M}^{\dagger m_M}|0\rangle = \prod_{k=1}^{M} n_k!\delta_{m_k\tilde{m}_k}$$

We call the operators $\tilde{a}_i$ the bosonic duals of the operators $a_i^\dagger$.

The transition number operators N_{ij}, the partial number operators N_i and the total number operator N can now be expressed as

$$\begin{aligned} N_{ij} &= a_i^\dagger \tilde{a}_j, \quad \forall (i,j), \\ N_i &= a_i^\dagger \tilde{a}_i, \quad \forall i, \\ N &= \sum_{i=1}^{M} a_i^\dagger \tilde{a}_i, \quad \forall i. \end{aligned}$$

From the expression for N_{ij} we obtain $\tilde{a}_j$ and vice versa. Symbolically,

$$\tilde{a}_j = a_j + \sum_{k\geq 1} a^{\dagger k} a^{k+1}.$$

For example, for $M = 3$ we find

$$\begin{aligned} \tilde{a}_i = & \frac{1}{(1+3\nu)}a_i + \frac{\nu}{(1+3\nu)}B_{0,1} \\ & - \frac{\nu}{4(1+3\nu)(2+3\nu)}b_{kji}^\dagger b_{kj}^2 \\ & - \frac{\nu(1+\nu)}{4(1+3\nu)^2(2+3\nu)}b_{kji}^\dagger b_{kji}^2 \\ & - \frac{\nu}{2(1+3\nu)^2(2+3\nu)}b_{kj}^\dagger b_{kj} b_{kji} + \cdots, \end{aligned}$$

where $b_{kji} = (a_k - a_i) + (a_j - a_i)$ and $b_{kj} = (a_k - a_j)$. Hence, we obtain new families of commuting operators $\tilde{a}_i(\nu)$, $i = 1, 2, \cdots, M$ and $\nu > -1/M$. They satisfy a new commutation relation:

$$[\tilde{a}_i(\nu), \tilde{a}_j{}^\dagger(\nu)] = \tilde{A}_{ij}(\nu),$$

and we call this the algebra $\tilde{\mathcal{A}}(\nu)$ dual to the algebra of (29). Of course, $[\tilde{a}_i(\mu), \tilde{a}_j(\nu)] \neq 0$ for $\nu \neq \mu$ (as $[a_i(\mu), a_j(\nu)] \neq 0$ for $\nu \neq \mu$).

The definition and structure of the algebra dual to a general algebra of the a_i and $a_i^\dagger$ operators is an interesting problem. Its physical meaning is connected with the construction of new models which are dual to the initial one.

4.5 Mapping to free Bose oscillators

It was found that the M-body Calogero model in the harmonic potential (28) could be mapped to M ordinary free Bose oscillators [24]. The mapping was performed in the coordinate space (not in the Fock space) and no restriction on ν was found or discussed. Since the whole Fock space (spanned by the monomials $a_{i_1}^{\dagger m_1}\cdots a_M^{\dagger m_M}|0\rangle$) for the M-body Calogero model with $\nu > -1/M$ is isomorphic to the Fock space of M free Bose oscillators with $\nu = 0$, we conclude that there must exist a regular mapping from $(a_i, a_i^\dagger)$ to $(b_i, b_i^\dagger)$ and vice versa. To ensure the existence of the mapping $a = \Psi(b, b^\dagger)$, the following relations have to be satisfied:

$$\begin{aligned} [a_i^\dagger, a_j^\dagger] &= [a_i, a_j] = 0, \\ [N_i, a_j^\dagger] &= \delta_{ij} a_i^\dagger, \quad \forall (i,j). \end{aligned} \tag{42}$$

The sufficient condition for the existence of the inverse real mapping $b = \Psi^{-1}(a, a^\dagger)$ is $\nu > -1/M$.

Our results on mappings can be generalized in the following way. If two algebras of operators, e.g. $(a_i, a_i^\dagger)$ and $(b_i, b_i^\dagger)$, have completely isomorphic Fock spaces (i.e. the same structure for all Gram matrices), then there exists a regular, real mapping from a_i to b_i and vice versa. If one Fock space is isomorphic with a subspace of the second Fock space, then there exists the mapping $a = \Psi(b, b^\dagger)$, but there is no inverse mapping. The construction of the mapping for Calogero operators $a_i, i = 1, \cdots, M$, see (29), will be considered in a separate publication.

5 Conclusion

In conclusion, we want to point out the main results of this paper. In Sect. 2 we have applied the general results of [9] to the Calogero–Vasiliev single-mode oscillator algebra (3), underlying the two-body Calogero model. We have expressed the number operator N, see (11), and the exchange operator K, see (10), as an infinite series in the creation and annihilation operators and have recursively calculated the coefficients of expansion. We have found the mapping (13) from Calogero–Vasiliev oscillators $(a, a^\dagger)$ to the Bose oscillators $(b, b^\dagger)$. The mapping has the form of an infinite series in $(b, b^\dagger)$. Finally, we have defined new operators $(\tilde{a}, \tilde{a}^\dagger)$ which are dual to the operators $(a, a^\dagger)$ in the sense that $[\tilde{a}, a^\dagger] = 1$. We have found a connection between the operators a, $\tilde{a}$ and b; see (12), (15) and (16).

Section 3 is devoted to the generalization of the single-mode oscillator algebras to the multi-mode case. We have discussed two distinct classes of multi-mode oscillator algebras:

(i) permutation invariant algebras with hermitean number operators and
(ii) permutation invariant algebras with non-hermitean number operators.

The class (ii) has not been discussed previously and in Sect. 3.2 several new results for those algebras were given. Both classes have been treated in three completely equivalent ways, first proposed in [7]. In the analysis, we have used concepts of a normally ordered expansion (18), the action of annihilation operators on the states in Fock space (19) and the notion of Gram matrices of scalar products in Fock space (20). We have found the general structure of the number operators, (23) and (24), and the exchange operators, (25)–(27). The results of this section have been applied in Sect. 4 to the S_M-extended Heisenberg algebra (29), underlying the M-body Calogero model. While the previous analyses of this algebra were performed mainly on the symmetric (or antisymmetric) subspace of the whole Fock space [6], here we have analyzed the whole Fock space of states. Our main results are the following.

We have rewritten the S_M-extended Heisenberg algebra in the form of the (generalized) triple operator algebra (31). This is a generalization of the known result for the single-mode case (3). Then, we have found the action of annihilation operators on the monomials in Fock space (35). Using this, we have calculated one- and two-particle Gram matrices and discussed their structure and eigensystem. We have found that there exists a universal critical point in Fock space, given by $\nu = -1/M$, and all states in Fock space have positive norms for $\nu > -1/M$. As we commented in Remark 4, the same critical point exists for a large class of S_M-extended Heisenberg algebras. Then, we have proceeded with the construction of number operators and exchange operators. We have given explicit examples of the structure of these operators in the case of $M = 3$, (37)– (39). Generalizing the construction of the dual algebra from Sect. 2, we have defined a dual multi-mode algebra in terms of the operators $(\tilde{a}_i, \tilde{a}_i^\dagger)$, (40) and (41). With these operators, we have been able to write the number operators in a compact form. Finally, we have briefly discussed the existence of a mapping from the S_M-extended Heisenberg algebra (29) to Bose oscillators.

We note that the Calogero model has been related [25] to q-deformed quantum mechanics [26]. The ideas presented here may help in elucidating the connection between algebraic structures arising from the deformation of the phase space of ordinary quantum mechanics and Calogero-type models.

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A Appendix

Here, we give two examples of complete Gram matrices for $M = 2$ and $M = 3$ oscillators and two-particle states.

Example A.1. The Gram matrix for $M = 2$. The matrix is written in the basis $\{a_1^{\dagger 2}|0\rangle,\, a_2^{\dagger 2}|0\rangle,\, a_1^\dagger a_2^\dagger|0\rangle,\, a_2^\dagger a_1^\dagger|0\rangle\}$:

$$\begin{pmatrix} a & b & b & b \\ b & a & b & b \\ b & b & d & d \\ b & b & d & d \end{pmatrix},$$

where $a = 2 + 3\nu$, $b = -\nu$ and $d = 1 + \nu$.

Example A.2. The Gram matrix for $M = 3$. The matrix is written in the basis $\{a_1^{\dagger 2}|0\rangle,\, a_2^{\dagger 2}|0\rangle,\, a_3^{\dagger 2}|0\rangle,\, a_1^\dagger a_2^\dagger|0\rangle,\, a_1^\dagger a_3^\dagger|0\rangle,\, a_2^\dagger a_3^\dagger|0\rangle,\, a_2^\dagger a_1^\dagger|0\rangle,\, a_3^\dagger a_1^\dagger|0\rangle,\, a_3^\dagger a_2^\dagger|0\rangle\}$.

$$\begin{pmatrix} a & b & b & b & b & c & b & b & c \\ b & a & b & b & c & b & b & c & b \\ b & b & a & c & b & b & c & b & b \\ b & b & c & d & b & b & d & b & b \\ b & c & b & b & d & b & b & d & b \\ c & b & b & b & b & d & b & b & d \\ b & b & c & d & b & b & d & b & b \\ b & c & b & b & d & b & b & d & b \\ c & b & b & b & b & d & b & b & d \end{pmatrix},$$

where $a = 2 + 6\nu + 2\nu^2$, $b = -\nu - \nu^2$, $c = 2\nu^2$ and $d = 1 + 3\nu + 2\nu^2$.

It is straightforward to write two-particle Gram matrices for any M. Non-zero matrix elements are of the type $((i, j, k, l) = 1, 2, \cdots, M)$

$$\begin{aligned} \langle 0|a_i^2 a_i^{\dagger 2}|0\rangle &\equiv a = [1 + \nu(M-1)][2 + \nu(M-1)] \\ &\quad - \nu^2(M-1), \\ \langle 0|a_i^2 a_j^{\dagger 2}|0\rangle &= \langle 0|a_i a_j a_i^{\dagger 2}|0\rangle = \langle 0|a_i a_j a_j^{\dagger 2}|0\rangle \\ &= \langle 0|a_i a_j a_k^\dagger a_j^\dagger|0\rangle = \langle 0|a_i a_j a_i^\dagger a_k^\dagger|0\rangle \\ &\equiv b = -\nu - \nu^2(M-2), \\ \langle 0|a_i a_j a_k^{\dagger 2}|0\rangle &= \langle 0|a_i a_j a_k^\dagger a_l^\dagger|0\rangle \equiv c = 2\nu^2, \\ \langle 0|a_i a_j a_i^\dagger a_j^\dagger|0\rangle &\equiv d = [1 + \nu(M-1)][1 + \nu(M-2)]. \end{aligned}$$

It is understood that different indices are not equal.

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